Group actions and cohomology

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Please ask questions!

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Part I

Introduction

What is the question?

Suppose an algebraic group G acts on a projective variety X. We are interested in what effect this has on the cohomology

$$H^*(X) = H^*(X, \mathbb{R}).$$

Clearly, we should assume that the action is faithful.
For example, we can always take $G = \mathrm{Aut}^0(X)$, the connected component of the automorphism group containing the identity.

General idea: $H^*(X) \cong H^*(G) \otimes V$

But this cannot be true in general...

The group $\mathbb{C}^\times$ acts on $\mathbb{P}^1$, but $H^1(\mathbb{C}^\times) \cong \mathbb{R}$ and $H^1(\mathbb{P}^1) = 0$.

Example: abelian varieties

When an abelian variety A acts faithfully on a projective variety, every orbit is a finite quotient of A:

- If G acts faithfully on X, the stabilizer $\mathrm{Stab}_x(G)$ of each point $x \in X$ is linear algebraic.
- Why? Consider the action on $\mathscr{O}_{X,x}/\mathfrak{m}_x^n$ for $n \gg 0$.
- So $\mathrm{Stab}_x(A)$ is affine and proper, hence finite.

In fact, the quotient $[X/A]$ exists as a stack, and

$$H^*(X) \cong H^*(A) \otimes H^*\Big([X/A]\Big).$$

By contrast:

Borel: The action of a solvable linear algebraic group on a projective variety always has a fixed point.

Chevalley's theorem

Algebraic groups come in two kinds:

- proper: abelian varieties
- affine: linear algebraic groups, e.g. $\mathrm{GL}_n(\mathbb{C})$

Chevalley: Every connected algebraic group G is an extension

$$1 \to L \to G \to A \to 1$$

of an abelian variety A by a linear algebraic group L.

Example: Let A be an abelian variety, $L \in \mathrm{Pic}^0(A)$. Then

$$1 \to \mathbb{C}^\times \to L^\times \to A \to 1,$$

where $L^\times$ is the line bundle with the zero section removed.

Example: homogeneous varieties

Suppose G acts faithfully on a smooth projective variety X. From Chevalley's theorem, we have $1 \to L \to G \to A \to 1$.

- The Albanese variety $\mathrm{Alb}(X)$ is an abelian variety.
- The action induces a morphism $G \to \mathrm{Alb}(X)$.
- L is rational, so $L \to \mathrm{Alb}(X)$ is trivial.
- The induced morphism $A \to \mathrm{Alb}(X)$ has finite kernel.

Borel-Remmert: If X is homogeneous, then

$$X \cong \mathrm{Alb}(X) \times F,$$

where F is homogeneous and rational. Therefore

$$H^*(X) \cong H^*\Big(\mathrm{Alb}(X)\Big) \otimes H^*(F).$$

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General idea: $H^*(X) \cong H^*(A) \otimes V$

Outline:

1. Cohomology of algebraic groups
2. Group actions and comodules
3. Main theorem and proof

Part II

Cohomology of algebraic groups

Hopf algebras

Let G be a connected algebraic group. The cohomology

$$H^*(G) = \bigoplus_{k \in \mathbb{Z}} H^k(G)$$

is a (commutative) graded $\mathbb{R}$-algebra with unit:

- The usual (cup) product

  $$H^*(G) \otimes H^*(G) \to H^*(G)$$

  comes from pullback along the diagonal $G \to G \times G$.
- One has $H^0(G) = \mathbb{R} \cdot 1$ by pulling back along $G \to \mathrm{pt}$.

Hopf algebras

The cohomology $H^*(G)$ is also a coalgebra:

- Pullback along $m\colon G \times G \to G$ gives a coproduct

$$\Delta\colon H^*(G) \to H^*(G) \otimes H^*(G).$$

- Pullback along $e\colon \mathrm{pt} \to G$ takes $1 \in H^0(G)$ to $1 \in \mathbb{R}$.

The coproduct Δ is a morphism of graded algebras; here

$$(\alpha_1 \otimes \alpha_2) \cdot (\alpha_3 \otimes \alpha_4) = (-1)^{\deg \alpha_2 \deg \alpha_3} (\alpha_1 \alpha_3) \otimes (\alpha_2 \alpha_4)$$

for the product on $H^*(G \times G) \cong H^*(G) \otimes H^*(G)$.

Hopf algebras

For $\alpha \in H^k(G)$, one has

$$\Delta(\alpha) = m^*(\alpha) \equiv 1 \otimes \alpha + \alpha \otimes 1 \mod \bigoplus_{i=1}^{k-1} H^i(G) \otimes H^{k-i}(G),$$

due to $e^*(1) = 1$ and the commutative diagram

$$\begin{tikzcd}
G \arrow[r, "e \times \mathrm{id}"] \arrow[dr, "\mathrm{id}"'] & G \times G \arrow[d, "m"] & G \arrow[l, "\mathrm{id} \times e"'] \arrow[dl, "\mathrm{id}"] \\
 & G &
\end{tikzcd}$$

This gives $H^*(G)$ the structure of a Hopf algebra.

Hopf algebras

Let H be a commutative graded $\mathbb{R}$-algebra with $H^0 = \mathbb{R} \cdot 1$. Then H is called a Hopf algebra if it has a coproduct

$$\Delta \colon H \to H \otimes H$$

that is a morphism of graded algebras, such that

$$\Delta(h) \equiv 1 \otimes h + h \otimes 1 \mod \bigoplus_{i=1}^{k-1} H^i \otimes H^{k-i}$$

for every $h \in H^k$.

Example: abelian varieties

Example: If A is an abelian variety, then $H^*(A) \cong \bigwedge H^1(A)$.

The coproduct is very explicit, too. One always has

$$\Delta(\alpha) = 1 \otimes \alpha + \alpha \otimes 1 \qquad \text{for } \alpha \in H^1(A).$$

Choose a basis $e_1, \ldots, e_{2g} \in H^1(A)$ and set

$$e_K = e_{k_1} \wedge \cdots \wedge e_{k_n} \qquad \text{for } K = \{k_1 < k_2 < \cdots < k_n\}.$$

Because Δ is a morphism of algebras, we get

$$\begin{aligned} \Delta(e_K) &= (1 \otimes e_{k_1} + e_{k_1} \otimes 1) \wedge \cdots \wedge (1 \otimes e_{k_n} + e_{k_n} \otimes 1) \\ &= \sum_{I \cup J = K} \operatorname{sgn}(I, J) \cdot e_I \otimes e_J, \end{aligned}$$

where $e_I \wedge e_J = \operatorname{sgn}(I, J) \cdot e_{I \cup J}$ for disjoint I, J.

Primitive elements

Let H be a Hopf algebra. An element $h \in H^k$ is primitive if

$$\Delta(h) = 1 \otimes h + h \otimes 1.$$

By definition, all of H^1 is primitive.

The homology of an algebraic group G is also a Hopf algebra, which is dual to $H^*(G)$. The subspace of primitive elements in $H_k(G, \mathbb{R})$ is spanned by the image of the Hurewicz map

$$\pi_k(G, e) \to H_k(G, \mathbb{Z}).$$

This is a result by Milnor and Moore.

The Milnor-Moore theorem

The structure of Hopf algebras is very simple.

Milnor-Moore: Any Hopf algebra is isomorphic to the wedge algebra on the graded vector space of its primitive elements.

If G is commutative and connected, then $G \cong \mathbb{C}^n/\Gamma$, where $\Gamma \cong \pi_1(G, e)$. The Milnor-Moore theorem says that

$$H^*(G) \cong \bigwedge H^1(G, \mathbb{R})$$

is the wedge algebra on $H^1(G, \mathbb{R}) \cong \mathrm{Hom}_{\mathbb{Z}}(\Gamma, \mathbb{R})$.

The abelian part of the cohomology

Let $1 \to L \to G \to A \to 1$ be as in Chevalley's theorem.

We get a short exact sequence of mixed Hodge structures

$$0 \to H^1(A) \to H^1(G) \to H^1(L) \to 0.$$

Deligne showed that $H^1(L)$ has weight 2 and type $(1,1)$.
Moreover, there is a unique decomposition

$$H^1(G) \cong H^1(A) \oplus H^1(L)$$

as mixed Hodge structures (over $\mathbb{R}$).

Conclusion: We have a surjective morphism of Hopf algebras

$$H^*(G) \to \bigwedge H^1(G) \to \bigwedge H^1(A) \to H^*(A),$$

compatible with mixed Hodge structures (over $\mathbb{R}$).

Part III

Group actions and comodules

Comodules

Suppose an algebraic group G acts on a projective variety X. Pulling back along the action

$$a\colon G \times X \to X$$

gives a morphism of graded $\mathbb{R}$-algebras

$$\delta\colon H^*(X) \to H^*(G) \otimes H^*(X).$$

From the commutative diagram

$$\begin{tikzcd} X \arrow[r, "{(e,\mathrm{id})}"] & G \times X \arrow[r, "a"] & X \end{tikzcd}$$

we see that any $m \in H^k(X)$ satisfies

$$\delta(m) \equiv 1 \otimes m \mod \bigoplus_{i=1}^{k} H^i(G) \otimes H^{k-i}(X).$$

Comodules

Moreover, the diagram

$$\begin{tikzcd}
G \times G \times X \arrow[r, "\mathrm{id} \times a"] \arrow[d, "m \times \mathrm{id}"] & G \times X \arrow[d, "a"] \\
G \times X \arrow[r, "a"] & X
\end{tikzcd}$$

is commutative; on cohomology, this turns into

$$\begin{tikzcd}
H^*(G) \otimes H^*(G) \otimes H^*(X) & H^*(G) \otimes H^*(X) \arrow[l, "\mathrm{id} \otimes \delta"'] \\
H^*(G) \otimes H^*(X) \arrow[u, "\Delta \otimes \mathrm{id}"] & H^*(X) \arrow[l, "\delta"'] \arrow[u, "\delta"]
\end{tikzcd}$$

So $H^*(X)$ is a comodule over the Hopf algebra $H^*(G)$.

Comodules

Let H be a Hopf algebra. A comodule is a graded $\mathbb{R}$-vector space M together with a morphism

$$\delta\colon M \to H \otimes M$$

such that, for every $m \in M^k$, one has

$$\delta(m) \equiv 1 \otimes m \mod \bigoplus_{i=1}^{k} H^i \otimes M^{k-i},$$

and such that the following diagram is commutative:

$$\begin{tikzcd}
H \otimes H \otimes M & H \otimes M \arrow[l, "\mathrm{id} \otimes \delta"'] \\
H \otimes M \arrow[u, "\Delta \otimes \mathrm{id}"'] & M \arrow[l, "\delta"'] \arrow[u, "\delta"']
\end{tikzcd}$$

Free comodules

A comodule M is free if there is an isomorphism

$$\phi\colon M \to H \otimes V$$

for a graded $\mathbb{R}$-vector space V, such that

$$\begin{tikzcd}
H \otimes H \otimes V & H \otimes V \arrow[l, "\Delta\otimes\mathrm{id}"'] \\
H \otimes M \arrow[u, "\mathrm{id}\otimes\phi"'] & M \arrow[l, "\delta"'] \arrow[u, "\phi"']
\end{tikzcd}$$

commutes.

In fact, V must be the subspace of primitive elements:

$$V = \left\{ m \in M \;\middle|\; \delta(m) = 1 \otimes m \right\}$$

Part IV

Main theorem and proof

Main theorem

Suppose an algebraic group G acts on a projective variety X.

- $H^*(X)$ is a comodule over the Hopf algebra $H^*(G)$.
- By Chevalley's theorem, we have

  $$1 \to L \to G \to A \to 1$$

  with A an abelian variety and L a linear algebraic group.
- We have a surjective morphism of Hopf algebras

  $$H^*(G) \to H^*(A).$$

- So $H^*(X)$ is also a comodule over $H^*(A)$.

Main theorem

Our main theorem generalizes a result by Ngô.

Theorem (de Cataldo, Kim, Schnell)

Let G be an algebraic group, A its maximal proper quotient. If G acts faithfully on a projective variety X, then

$$H^*(X) \cong H^*(A) \otimes V$$

is free as a comodule over $H^*(A)$.

- Ngô's proof used reduction to finite fields.

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Let G be an algebraic group, A its maximal proper quotient. If G acts faithfully on a projective variety X, then

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is free as a comodule over $H^*(A)$.

- Ngô's proof used reduction to finite fields.
- Our proof uses Hodge theory.
- The result is also true for meromorphic actions on complex spaces that are compact and Kähler.

Outline of the proof

The proof has two parts:

1. A cohomological argument to relate $H^1(X)$ and $H^1(A)$.
2. A formal argument about the structure of $H^*(X)$.

The second part is in the style of Grothendieck and Deligne: once enough structure is in place, the result proves itself.

Part 1 of the proof

We may assume that G and X are connected.

The comodule structure on $H^*(X)$ gives us

$$\delta\colon H^1(X) \to H^1(X) \oplus H^1(G), \quad \delta(m) = \big(m, q(m)\big).$$

The mixed Hodge structure on $H^1(X)$ has weight ≤ 1, and so

$$q\colon H^1(X) \to H^1(G)$$

factors through $W_1 H^1(G) = H^1(A)$.

Lemma

The image of $q\colon H^1(X) \to H^1(G)$ *is equal to* $H^1(A)$.

Part 1 of the proof

Lemma

The image of $q\colon H^1(X) \to H^1(G)$ *is equal to* $H^1(A)$.

Let's prove this when X is smooth. The action induces

$$\begin{tikzcd} G \times X \arrow[r, "a"] \arrow[d] & X \arrow[d] \\ A \times \mathrm{Alb}(X) \arrow[r] & \mathrm{Alb}(X) \end{tikzcd}$$

and $A \to \mathrm{Alb}(X)$ has finite kernel. Therefore

$$H^1(X) \cong H^1\big(\mathrm{Alb}(X)\big) \to H^1(A)$$

must be surjective.

Part 1 of the proof (repeated)

We may assume that G and X are connected.

The comodule structure on $H^*(X)$ gives us

$$\delta\colon H^1(X) \to H^1(X) \oplus H^1(G), \quad \delta(m) = \big(m, q(m)\big).$$

The mixed Hodge structure on $H^1(X)$ has weight ≤ 1, and so

$$q\colon H^1(X) \to H^1(G)$$

factors through $W_1 H^1(G) = H^1(A)$.

Lemma

The image of $q\colon H^1(X) \to H^1(G)$ is equal to $H^1(A)$.

Part 2 of the proof

There is again a unique decomposition

$$H^1(X) \cong W_0 H^1(X) \oplus \mathrm{gr}_1^W H^1(X)$$

as mixed Hodge structures (over $\mathbb{R}$).

Using the lemma, we can choose a splitting

$$s\colon H^1(A) \to H^1(X)$$

with the property that $q \circ s = \mathrm{id}$.

Via cup product, this gives $H^*(X)$ the additional structure of a module over the graded algebra

$$H^*(A) = \bigwedge H^1(A).$$

The rest of the proof is formal.

Part 2 of the proof

Suppose that M is both a module and a comodule over

$$H = \bigwedge H^1,$$

in such a way that $\delta \colon M \to H \otimes M$ satisfies

$$\delta(h \cdot m) = \Delta(h) \cdot \delta(m).$$

Then M is free as a comodule over H.

- Consider the subspace of primitive elements

$$V = \left\{ m \in M \;\middle|\; \delta(m) = 1 \otimes m \right\}.$$

- Using the module structure, we get $H \otimes V \to M$.
- Then check that this is an isomorphism.

Thank you!